OPTIMAL ESTIMATES IN THERMAL MEASUREMENTS

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Presented is an analysis of the use of a Kalman filter to improve the accuracy of thermal-measurement systems and to increase the information which can be obtained from them.

Thermal measurements include determinations of both (a) the temperatures of solid and gaseous media and (b) thermal properties which by their nature are only indirectly related to temperatures, e.g., heat fluxes, radiation fluxes, thermophysical characteristics of materials, surface heat-transfer characteristics, etc. In both cases the measured properties are found (calculated) from the directly detected temperatures of temperature pickups in the media under study. The information about these media includes, in addition to the useful signal, random errors or noise. Problems of this type are extremely complicated, falling in the class of "incorrectly formulated inverse problems" [1]. There are essentially no structural methods for solving incorrectly formulated inverse problems for applications to thermal measurements; as a result, there is room for a general improvement in the accuracy and information content of most thermal-measurement systems.

In the present paper we propose the use in this connection of certain cybernetic methods, in particular, optimal estimates and identification through the use of a numerical Kalman filter. These methods are based on an analysis of the dynamics of the thermal-measurement system in its state space [3].

1. Mathematical Model of the System

The dynamics of a thermal-measurement system, which usually includes one or several temperature pickups, is described by actual differential equations and is represented as the temperature field in some multiply connected region. It has been shown [4] that at each time τ this field can be represented by the temperatures of a finite number of points N, which form a state vector of the system:

$$\vec{X}(\tau) = |x_1 x_2 \ldots x_i \ldots x_N|^T.$$

Its values at successive discrete times $k \cdot \Delta\tau$ and $(k+1)\Delta\tau$ are related by [3, 6]

$$\vec{X}(\tau)\big|_{\tau=(k+1)\Delta\tau} = \Phi_{k+1,k} \cdot \vec{X}(\tau)\big|_{\tau=k\cdot\Delta\tau}. \quad (1)$$

The set of vectors $\vec{X}(\tau)$ forms the state space of the system. The dynamics of the system can be described approximately by N first-order ordinary differential equations in the components x_i of the state vector. In vector-matrix form, these equations can be written

$$\frac{d\vec{X}(\tau)}{d\tau} = \dot{\vec{X}}(\tau) = F(\tau)\vec{X}(\tau) + G(\tau)\cdot\vec{U}(\tau). \quad (2)$$

In the more general case in which the thermophysical characteristics of the material depend on the temperature and in the case of nonlinear boundary conditions the system is described by the nonlinear equation

$$\dot{\vec{X}}(\tau) = f_1(\vec{X}(\tau), \vec{G}(\vec{X}), \vec{U}(\tau), \tau). \quad (3)$$

Estimates of the error of a differential-difference approximation of this type can be found in papers on the finite-difference methods for solving boundary-value problems, in particular, by the straight-line method [5]. Some practical examples of this method and the reasoning behind it are given in [4].

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Equations (2) or (3), which depend on the nature of the measurements in some manner, incorporate the measured thermal properties. For example, the temperatures of a solid are usually the components of the state vector $\vec{X}(\tau)$. The temperature of a gaseous medium flowing around the thermal-measurement system is part of the control agent $\vec{U}(\tau)$. The surface heat-transfer coefficient $F(\tau)$ appears in both the control agent and the feedback matrix $F(\tau)$, while the thermophysical characteristics of the material appear in the feedback matrix, etc.

Describing the system in this manner, we can analyze its behavior (reaction) upon changes in the unknown thermal properties; in other words, we can solve the direct heat-conduction problem. It is extremely convenient to follow this approach in the state space of the system, based on a calculation of the transfer matrix $\Phi_{k+1,k}$ from the feedback matrix $F(\tau)$. If $\Phi_{k+1,k}$ is known, we can find a solution in the temporal region in accordance with Eq. (1). This approach permits us to treat both steady-state and transient systems for arbitrary (not necessarily vanishing) initial conditions. Furthermore, it is possible to carry out a more general analysis of the system: its controllability, the observability, the choice of an optimum dimensionality for the model, etc. [2, 4].

However, we are primarily interested in the possibility of applying to such a system the methods of optimal filtering, estimates, and identification for solving the inverse problem.

In this case we must supplement the basic equation for the system, (2) or (3), with the following equation for the observation vector:

$$\vec{Y}(\tau) = H_1(\tau) \cdot \vec{X}(\tau) + \vec{W}(\tau). \tag{4}$$

The components of y_i ($i = 1, 2, \ldots, N$) of the observation vector are the measurements (observations) of the components x_i of the state vector. Here $\vec{Y}(\tau)$ usually directly incorporates the recorded indications of the thermal pickups (thermocouples or resistance thermometers) in the system. The noise vector $\vec{W}(\tau)$, which is unavoidably present in the measurements, is assumed to be white Gaussian noise with a zero expected value, $E[\vec{W}(\tau)] = 0$, and a covariance matrix

$$\operatorname{cov}[\vec{W}(\tau) \cdot \vec{W}^T(\tau)] = E[\vec{W}(\tau) \cdot \vec{W}^T(\tau)] = N = \operatorname{diag}|n_1 n_2 \ldots n_N|.$$

2. Optimal Estimates and Identification

The mathematical model of the system can thus be described by Eqs. (2) and (4).

As was mentioned above, among the components of the state vector, parameters, or control agents are unknown thermal properties which must be determined from the values of the observation vector $\vec{Y}_k = \vec{Y}(\tau)|_{\tau = k\Delta\tau}$, usually specified at discrete times.

In the terminology of general systems theory this problem can be solved only through an optimal estimate of the state vector, in the case in which the measured property is a component of this vector, or through identification, in which case the measured property is included in the feedback matrix or in the control agent of the system.

In the former case, direct use can be made of the (discrete) numerical algorithm of optimal sequential filtering proposed by Kalman [6]. In the latter case, which is more common, it is first necessary to expand the state vector of the system by incorporating in it the measured properties. We will discuss this formal operation in more detail and show that it is feasible in many cases of practical importance.

For example, when the property to be identified, α_j, is constant ($\alpha_j = \text{const}$), an additional component $x_j = \alpha_j$ is incorporated in the state vector, and we supplement system (2) or (3) with the equation

$$\dot{x}_j = 0. \tag{5}$$

If the property to be identified is a linear function of the time, $\alpha_j = \alpha_0 + \alpha_1 \tau$, then by denoting $x_j = \alpha_j$ and $x_{j+1} = \alpha_1$ and carrying out two successive differentiations of x_j we find an additional system of two equations:

$$\dot{x}_j = x_{j+1}, \qquad \dot{x}_{j+1} = 0. \tag{6}$$

When system (2) or (3) is supplemented with Eqs. (6), the state vector expands by two components, x_j and x_{j+1}. In this manner we can describe unknown properties having various time dependences: exponential, sinusoidal, polynomial, and various combinations thereof [7].

If the properties to be identified are included in the matrix $F(\tau)$ as well as in the input-agent vector, as is the case for the heat-transfer coefficients and the thermophysical characteristics of the material, the expanded system becomes nonlinear and can be written

$$\dot{\vec{R}}(\tau) = f(\vec{R}(\tau)),$$

$$\vec{Y}_k = H\vec{R}_k + \vec{W}_k,$$

where

$$\vec{R}(\tau) = \begin{vmatrix} \vec{X}(\tau) \\ \vec{\alpha}(\tau) \end{vmatrix}. \tag{7}$$

The unknown-property vector $\vec{\alpha}(\tau)$ is of dimensionality $m \times 1$.

Now the expanded state vector $\vec{R}(\tau)$ of dimensionality $(N + m) \times 1$ of nonlinear system (7) can be subjected to a sequential optimal estimate. System (7) is usually linearized along some reference trajectory $\vec{R}^*(\tau)$; the reference trajectory itself is also estimated sequentially.

The algorithm of the discrete linear Kalman filter is applied to the linearized system, written in discrete form. This algorithm consists of the following sequence of operations on the vectors and matrices [6]:

$$\hat{\vec{R}}_{k+1|k+1} = \hat{\vec{R}}_{k+1|k} + K_{k+1}[\vec{Y}_{k+1} - H\hat{\vec{R}}_{k+1|k}], \tag{8}$$

$$\hat{\vec{R}}_{k+1|k} = \Phi_{k+1,k}\cdot\hat{\vec{R}}_{k|k}, \tag{9}$$

$$K_{k+1} = P_{k+1|k}H^T[HP_{k+1|k}H^T + N], \tag{10}$$

$$P_{k+1|k} = \Phi_{k+1,k}P_{k|k}\Phi^T_{k+1,k}, \tag{11}$$

$$P_{k+1|k} = P_{k+1|k} - K_{k+1}HP_{k+1|k}, \tag{12}$$

where

$$\Phi_{k+1,k} \approx \exp F_{k+1}\cdot\Delta\tau = I + F_{k+1}\cdot\Delta\tau + F^2_{k+1}\frac{(\Delta\tau)^2}{2!} + \ldots$$

$$F_{k+1} = \left.\frac{\partial f(\vec{R}(\tau))}{\partial \vec{R}(\tau)}\right|_{\vec{R}(\tau)=\vec{R}^*_k},$$

$$\vec{R}^*_k \triangleq \vec{R}^*(\tau)|_{\tau=k\cdot\Delta\tau}.$$

The quadratic feedback matrix F_{k+1} of the linearized system is of dimensionality $N + m$. The equations for its elements include estimates of the state vector $\hat{\vec{X}}_{k|k}$ and the vector of unknowns $\hat{\vec{\alpha}}_{k|k}$ obtained at the preceding time.

It follows from this algorithm that the new expanded state vector $\hat{\vec{R}}_{k+1|k+1}$ is equal to [Eq. (8)] to the sum of its prediction $\hat{\vec{R}}_{k+1|k}$, extrapolated through Eq. (9) for the dynamics of the system on the basis of the "old" estimate $\hat{\vec{R}}_{k|k}$ and the weighted difference between the real measurement, $\vec{Y}_{k+1}$, and the prediction of this measurement.

The sequential procedure for estimating the state vector $\hat{\vec{X}}_{k|k}$ and identifying the vector of unknown parameters $\hat{\vec{\alpha}}_{k|k}$ is as follows.

For the initial time, $k = 0$, we specify an initial expanded state vector $\hat{\vec{R}}_{0|0}$, consisting of the initial vectors $\hat{\vec{X}}_{0|0}$ and $\hat{\vec{\alpha}}_{0|0}$. Obviously, only those components $\hat{x}_i(0)$ of the state vector which appear in the observation vector of the system, $\vec{Y}_0$, can be written with adequate accuracy. The other components of the state vector $\hat{\vec{X}}_{0|0}$ and the vector of unknowns $\hat{\vec{\alpha}}_{0|0}$ are specified with large errors, governed by the level of our a priori or estimated knowledge. The dispersion corresponding to the probability of the a priori information is reflected in the choice of elements of the covariance matrix $P_{0|0}$ for the errors in the initial estimates [6]. The values of $\vec{R}_{0|0}$ and $P_{0|0}$ and the dispersions $n_1 = n_2 = \ldots = n_N$, which are a measure of the measurement noise, serve as the basis for the first calculation step. Here the components $\hat{x}_i(0)$ and $\hat{\alpha}_i(0)$ of the initial expanded state vector $\vec{R}_{0|0}$ are used to calculate the feedback matrix of the linearized

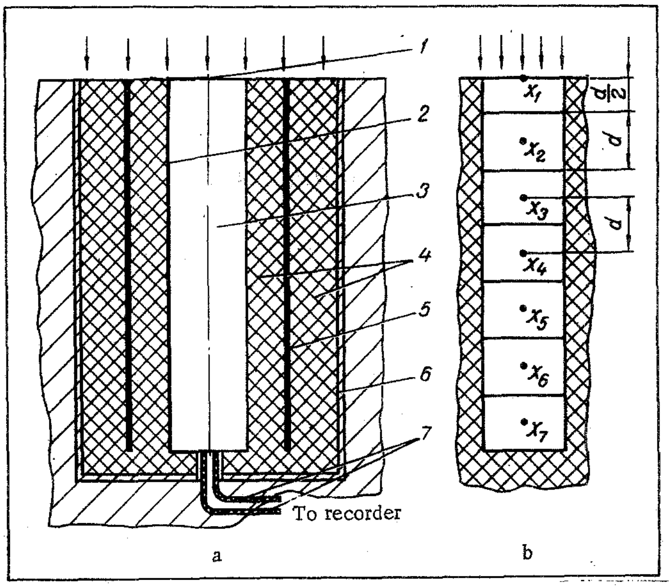

Fig. 1. a) Construction of the heat-flux pickup; b) model of this pickup. 1) Junction of film thermocouple; 2) film thermoelectrode; 3) thermal pickup; 4) thermal insulation; 5) protective cylinder; 6) housing; 7) leads.

system, F_1. Using the values of the observation vector $\vec{Y}_1$ at our disposal along with Eqs. (8)-(12), we determine estimates of the expanded state vector $\hat{\vec{R}}_{1|1}$ and the covariance matrix $\hat{P}_{1|1}$ of the errors of these estimates. Then the procedure is repeated, with a calculation of $\vec{R}_{2|2}$ and $P_{2|2}$, $\hat{\vec{R}}_{3|3}$ and $P_{3|3}$, etc.

In the course of the calculations the vector of unknowns is constantly refined; at some time the estimates $\overleftarrow{\alpha}_{k|k}$ converge to definite values, and the corresponding diagonal matrix elements $P_{k|k}$ become extremely small. It must be noted here that the diagonal matrix elements $P_{k,k}$ are the dispersions which are a measure of the accuracy in the estimates found for the unknowns in the k-th calculation step [6].

Accordingly, the inverse problem for a thermal-measurement system is solved through the use of the Kalman filter in the following manner:

1) A mathematical model of the system is constructed which includes both the directly measurable properties, as well as the thermal properties, which are to be determined.

2) In a manner governed by the nature of the measurements, the state vector of the system is expanded by the quantities which are to be determined.

3) As information becomes available there is a sequential estimate of the expanded state vector of the system and thus an identification of the unknown thermal properties.

This method has several advantages: First, the unknown thermal quantity found by solving the inverse problem, so that methodological errors can be essentially eliminated (the dynamic error due to heat transfer along the thermal pickup and so forth). Such errors usually reduce the measurement accuracy. The transient, nonlinear systems can be treated without important simplifications, and the noise present in the information does not affect the accuracy of the final results. Second, if some a priori information is available, it is possible to find all the thermal properties and the parameters which are reflected in the mathematical model of the thermal-measurement system.

To illustrate this discussion we consider an example.

Example

We are to determine the temperature of the medium, t_{me}, and the heat-transfer coefficient at the surface of the pickup, α.

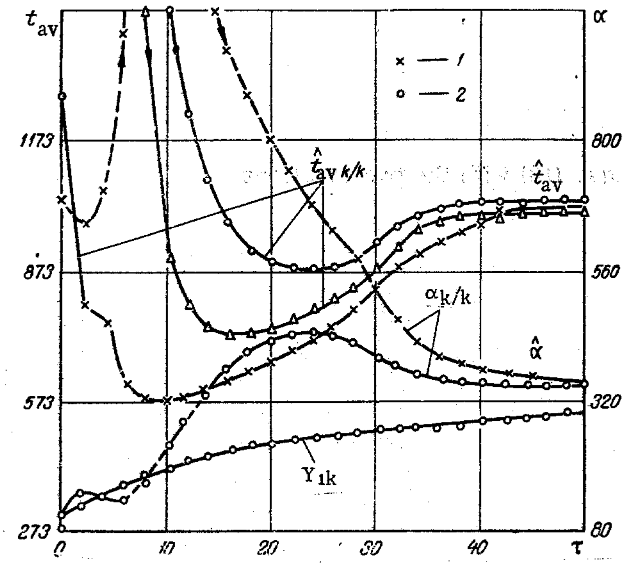

Fig. 2. Practical determination of the temperature of the medium, t_{me} (°K), and of the heat-transfer coefficient α [W/(m^2·deg)] at the end surface of the pickup. Also shown here is the behavior of the estimates $\hat{t}_{av\,k|k}$ (solid curve) and $\hat{\alpha}_{k|k}$ (dashed curve) for certain versions of the initial estimates. 1) Version 1; 2) version 3. Here Y_{1k} is the directly measured temperature of the end of the pickup. Here τ is in seconds.

We use the procedure outlined above to treat the real problem of determining the flow properties of the combustion products of kerosene from direct measurements of the temperature at the end surface of a pickup in the flow.

The pickup (Fig. 1a) consists of a cylindrical rod 4 mm in diameter and 39 mm long, enclosed in a hollow protective cylinder, which is filled with cast alundrum. The pickup itself and the protective cylinder are made of type ZhS6-K alloy. On the basis of handbook data we assume that the thermal diffusivity of the material is a linear function of the temperature, $a = a_0 + \varkappa x$ [$a_0 = 2.40 \cdot 10^{-6}$ m^2/sec, $\varkappa = 4.80 \cdot 10^{-9}$ m^2/(sec·deg)], and we assume $c\gamma = 3.4 \cdot 10^6$ J/(m^3·deg). The temperature of the end of the pickup is measured by a "semisynthetic" platinum surface film thermocouple, like that described in [8]. The joint of the thermocouple — the point at which the leads (made of wire) to the film and to the pickup material are connected — is at the thermally insulated end of the pickup, where the temperature is also monitored.

The mathematical model, in correspondence with the discussion in [4], is obtained by partitioning the pickup into N = 7 regions or blocks (Fig. 1b). This model is described by

$$\dot{x}_1 = -\left(\frac{2a_0}{d^2} + \frac{2}{c\gamma d}\alpha\right)x_1 + \frac{2a_0}{d^2}x_2 + \frac{\varkappa}{d^2}(x_2^2 - x_1^2) + \frac{2}{c\gamma d}\alpha t_{av},$$

$$\dot{x}_2 = \frac{a_0}{d^2}x_1 - \frac{2a_0}{d^2}x_2 + \frac{a_0}{d^2}x_3 + \frac{\varkappa}{2d^2}(x_1^2 - 2x_2^2 + x_3^2),$$

$$\dots\dots\dots\dots\dots\dots\dots\dots$$

$$\dot{x}_i = \frac{a_0}{d^2}x_{i-1} - \frac{2a_0}{d^2}x_i + \frac{a_0}{d^2}x_{i+1} + \frac{\varkappa}{2d^2}(x_{i-1}^2 - 2x_i^2 + x_{i+1}^2), \qquad (13)$$

$$\dots\dots\dots\dots\dots\dots\dots\dots$$

$$\dot{x}_7 = \frac{a_0}{d^2}x_6 - \frac{a_0}{d^2}x_7 + \frac{\varkappa}{d^2}(x_6^2 - x_7^2).$$

In writing Eqs. (13) we assumed that in the heat transfer from block i to block j the thermal diffusivity is that corresponding to the average block temperatures; i.e., $a_{i-j} = a_0 + \varkappa[(x_i + x_j)/2]$.

In this case the observation vector $\vec{Y}_k$ is a scalar, given by

$$\vec{Y}_k = x_{1k} + w_{1k}.$$

We supplement the state vector $\vec{X}(\tau)$ of the system with the vector of quantities which are to be determined:

$$\vec{\alpha} = |(\alpha t_{av})\, \alpha|.$$

Working from the assumed constancy of the heating conditions at the end of the pickup (t_{av} = const, α = const), we supplement Eqs. (13) with the two equations

$$(\alpha \dot{t}_{av}) = 0 \text{ and } \dot{\alpha} = 0.$$

Accordingly, in this case we find the equation of the expanded system in form (7), where $\vec{R}(\tau) = \begin{vmatrix} \vec{X}(\tau) \\ \vec{\alpha}(\tau) \end{vmatrix}$ — (9 × 1) is the expanded state vector, f is a nine-dimensional nonlinear vector function, and H = |10 ... 0| — (1 × 9) is the observation matrix.

The expanded state vector $\vec{R}(\tau)$ is to be identified through sequential estimates on the basis of the algorithm described above.

Figure 2 shows the results of a practical identification of α and t_{me} through the use of the Kalman-filter algorithm (8)-(12). As the initial values we specify the accurate values of the state vector $\hat{\vec{X}}_{0|0}$ of the system, working from the essentially uniform initial temperature distribution which prevails over the thermal pickup [$x_1 = x_2 = \ldots = x_7 = x(0)$]. We assume the following combinations of arbitrarily selected values of the unknowns α and t_{me}: 1) α = 720 W/m^2, t_{me} = 1273°K; 2) 110, 573; 3) 110, 2273; 4) 110, 1273; 5) 720, 573; and several other combinations. In all cases the identification procedure converges within ±2% of certain values of the quantities to be identified. These values are then adopted as the actual values: α = 355 W/m^2 and t_{me} = 1010°K.

Figure 2 shows the behavior of the estimates $\hat{\alpha}_{k|k}$ and $\hat{t}_{av\,k|k}$ as they approach their steady-state (actual) values; this behavior characterizes the identification process.

Direct control measurements of t_{me} carried out with a Chromel–Alumel thermocouple 0.1 mm in diameter show that the flow temperature varies from 993 to 1038°K, depending on the distance from the end of the pickup. Since it is some average of the flow temperature over the thickness which is identified, the results were judged favorable.

We have thus shown that the theory of optimal filtering, estimates, and identification can be used to obtain optimal estimates of measured thermal properties. As an example we reported the practical application of this procedure for determining the temperature of a gas flow and for determining the heat-transfer coefficient along the surface of a pickup (a wall).

NOTATION

T	is the transposed vector or matrix;
$\Delta\tau$, sec	is the time step;
$\Phi_{k+1,k}$	is the transfer matrix of the system;
$F(\tau)$ — (N × N)	is the feedback matrix;
$G(\tau)$ — (N × p)	is the input-agent matrix;
$\vec{U}(\tau)$ — (p × 1)	is the input-agent vector;
f_1 and f	are the nonlinear vector functions;
$H(\tau)$ — (r × N)	is the observation matrix;
$\vec{R}(\tau)$	is the expanded state vector;
I	is the unit matrix;
$\hat{\vec{X}}_{k+1|k+1}$ and $\hat{\vec{R}}_{k+1|k+1}$	are the estimates of the state vector and expanded state vector [the double subscript is used to give the time for which the estimate is made (the first subscript) and the observation time (the second)];
$\hat{\vec{R}}_{k+1|k}$	is the prediction of the expanded state vector to the time k + 1;
K_{k+1}	is the weight matrix of the filter;
$P_{k+1|k+1}$	is the covariance matrix of the errors of the estimate;
$P_{k+1|k}$	is the covariance matrix of the errors of the prediction;
G, J/(kg · deg) and γ, kg/m^3	are the specific heat capacity and density of the material.

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